TEMPERATURE FIELD OF TWO-DIMENSIONAL REGIONS WITH AXIAL SYMMETRY

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The problem of the approximate analytic description of the steady temperature field of two-dimensional regions with axial symmetry is examined.

Even in the simplest cases of homogeneous media and uniformly distributed sources, the integration of the equation of heat conduction for two-dimensional regions with a complex boundary usually involves falling back on either numerical or analog techniques. However, approximate analytic solutions may be preferable, for example, from the standpoint of establishing their parametric dependence.

Particularly attractive is the Ritz method, in which integration of the differential equation is replaced by the problem of finding the function that minimizes the functional corresponding to the starting equation.

Thus, we consider a two-dimensional axisymmetric figure, the cross section of an infinite homogeneous cylindrical body with uniformly distributed heat sources. We assume that heat exchange with the surrounding medium, whose temperature is taken as zero, is realized in accordance with Newton's law

$$\nabla_n T + \frac{\alpha}{\lambda} T|_\Gamma = 0, \tag{1}$$

where Γ is the contour bounding the region in question.

Introducing the notation

$$u(r,\ \varphi) = \frac{T(r,\ \varphi)}{T^*}, \tag{2}$$

where T^* is the temperature at the center of the rod, we write the approximate solution of the problem in the form

$$u(r,\ \varphi) \approx \sum_{k=0}^{n} C_k u_k(r,\ \varphi). \tag{3}$$

For the heat conduction equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = -\frac{q_v}{\lambda T^*} \tag{4}$$

the functional of the problem has the form

$$I_n(u) = \int_0^{2\pi} d\varphi \int_0^{f(\varphi)} dr\, r\left\{\left(\sum_{k=0}^{n} C_k \frac{\partial u_k}{\partial r}\right)^2 + \frac{1}{r^2}\left(\sum_{k=0}^{n} C_k \frac{\partial u_k}{\partial \varphi}\right)^2 - \frac{2q_v}{\lambda T^*}\sum_{k=0}^{n} C_k u_k\right\}, \tag{5}$$

where the coefficients C_k are determined by the system of equations (condition of minimum of the functional)

$$\sum_{k=0}^{n} C_k A_{jk} = \frac{q_v}{\lambda T^*} B_j, \quad j = 0,\ 1,\ 2,\ \ldots,\ n. \tag{6}$$

Here,

$$A_{jk} = \int_0^{2\pi} d\varphi \int_0^{f(\varphi)} W_{jk}(r,\ \varphi)\, r dr, \tag{7}$$

$$W_{jk}(r,\ \varphi) = \frac{\partial u_j}{\partial r}\frac{\partial u_k}{\partial r} + \frac{1}{r^2}\frac{\partial u_j}{\partial \varphi}\frac{\partial u_k}{\partial \varphi}, \tag{8}$$

$$B_j = \int_0^{2\pi} d\varphi \int_0^{f(\varphi)} u_j(r,\ \varphi)\, r dr. \tag{9}$$

The set of functions $\{u_k\}$ can be arbitrarily selected. It is only necessary that it be linearly independent and that, at the boundary of the region $r_b = f(\varphi)$, it satisfies the condition

$$\nabla_n u_k + \frac{\alpha}{\lambda} u_k|_\Gamma = 0. \tag{10}$$

The latter equation can be represented in the form

$$\frac{1}{\sqrt{1+\left(\frac{f'}{f}\right)^2}}\frac{\partial u_k}{\partial r} + \frac{1}{\sqrt{1+\left(\frac{f}{f'}\right)^2}}\frac{1}{r}\frac{\partial u_k}{\partial \varphi} = -\frac{\alpha(\varphi)}{\lambda}u_k. \tag{11}$$

From the integrals of this equation

$$H_1 = \ln\frac{r}{r_0} - \int \frac{f'}{f} d\varphi,$$

$$H_2 = \ln u_k + \frac{r}{p\lambda}\int \sqrt{1+\left(\frac{f}{f'}\right)^2}\, p\alpha\, d\varphi, \tag{12}$$

where $p(\varphi) = \exp\left\{\int \frac{f'}{f} d\varphi\right\}$ and r_0 is some characteristic dimension, we construct an expression for $u_k(r,\ \varphi)$. Any function of the form $\Phi(H_1, H_2) = 0$ is a solution of Eq. (11); i.e., for example, from the relation

$$H_2 - \exp\{2kH_1\} = 0,$$

$$u_k(r,\ \varphi) = \left(\frac{r}{r_0}\right)^{2k}\exp\left\{\frac{r}{p\lambda}\int\sqrt{1+\left(\frac{f}{f'}\right)^2} \times p\alpha\, d\varphi + 2k\int\frac{f'}{f}d\varphi\right\}. \tag{13}$$

Having determined the coefficients $\{C_k\}$ from the system of linear equations (6), we can find an expression for T*

$$T^* = -\frac{\frac{q_v}{2\lambda}\int_0^{2\pi} f^2(\varphi)\,d\varphi}{\int_0^{2\pi}\left[\frac{\partial u}{\partial r}f + \frac{f'}{f}\frac{\partial u}{\partial \varphi}\right]_{r=f(\varphi)} d\varphi}, \qquad (14)$$

which follows from the balance relation $-\lambda\oint_L \nabla_n T dl = q_v S$ for a height interval of unit length. Here, $S = \frac{1}{2}\int_0^{2\pi} f^2(\varphi)\,d\varphi$ is the area of the figure bounded by the curve Γ.

In the case of a boundary value problem with the homogeneous boundary condition $T|_\Gamma = 0$ the first approximation, suitable for figures approximating a circle, is of interest. The difficulties in selecting a system of trial functions can be eliminated by assuming symmetry of the contour bounding the plane section.

If, with this assumption, as the system of functions $\{u_k(r, \varphi)\}$, used to construct the solution in the form $u(r, \varphi) \approx \sum_{k=1}^{n} C_k u_k(r, \varphi)$, we take either

$$u_k(r, \varphi) = 1 - \left[\frac{r}{f(\varphi)}\right]^{2k} \qquad (15')$$

or

$$u_k(r, \varphi) = \left[1 - \frac{r^2}{f^2(\varphi)}\right]^k, \qquad (15'')$$

then the best approximation to the solution is $C_1 u_1(r, \varphi)$. Except for C_1 all the coefficients found from system (6) vanish upon substitution of $u_k(r, \varphi)$ in form (15) and thus

$$T^* \approx \frac{q_v}{4\lambda}\frac{\int_0^{2\pi} f^2(\varphi)\,d\varphi}{\int_0^{2\pi}\left[1 + \left(\frac{f'}{f}\right)^2\right]d\varphi}. \qquad (16)$$

In fact, in this case system (6), taking the form

$$\sum_{k=2}^{n}\frac{(j+1)k}{(j+k)}C_k = \frac{q_v}{4\lambda T^*}\frac{\int_0^{2\pi} f^2(\varphi)\,d\varphi}{\int_0^{2\pi}\left[1 + \left(\frac{f'}{f}\right)^2\right]d\varphi} - C_1, \qquad (17)$$

$$j = 1, 2, 3, \ldots, n,$$

has only the trivial solution $C_k = 0$, $k = 2, 3, 4, \ldots, n$, and, consequently, from the condition at the center of the rod $\sum_{k=1}^{n} C_k = 1$ we obtain expression (16).

However, the value of T* found from (16) gives too low a result as compared with expression (14), since it follows from the same balance relation, if the boundary gradient is replaced by its modulus.

Setting $u(r, \varphi) = 1 - r^2/f^2(\varphi)$ in (14), we obtain

$$T^* \approx \frac{q_v}{4\lambda}\frac{\int_0^{2\pi} f^2(\varphi)\,d\varphi}{\int_0^{2\pi}\left[1 - \left(\frac{f'}{f}\right)^2\right]d\varphi}. \qquad (18)$$

We present several examples of approximate expressions for T* obtained from (18).

1. Rod of elliptical cross section: $f(\varphi) = \sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}$

$$T^* \approx \frac{q_v}{2\lambda}\frac{a^2b^2}{(4ab - a^2 - b^2)}. \qquad (19)$$

2. Rod with a cruciform profile described by the equation $f(\varphi) = a + b\cos 4\varphi$

$$T^* \approx \frac{q_v}{4\lambda(1-\xi)}\left(a^2 + \frac{b^2}{2}\right), \qquad (20)$$

where

$$\xi = \frac{16\left(a^3 - ab^2 - \sqrt{(a^2-b^2)^3}\right)}{\sqrt{(a^2-b^2)^3}}.$$

3. Rod with a profile in the form of a regular polygon (with number of sides, n, and radius of circumscribed circle, a)

$$T^* \approx \frac{q_v a^2}{4\lambda}\frac{n\sin\frac{\pi}{n}\cos\frac{\pi}{n}}{2\pi - n\,\mathrm{tg}\,\frac{\pi}{n}}. \qquad (21)$$

For example, for a square (n = 4) this formula gives a result which differs from the exact expression

$$T^* = \frac{q_v a^2}{4\lambda}\left[1 + \frac{32}{\pi^3}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k+1)\,\mathrm{ch}\,(2k+1)\frac{\pi}{2}}\right] \qquad (22)$$

by a coefficient 0.88.

It is easy to see that when $f(\varphi)$ = const expression (18) gives the value for the temperature at the center of an infinite circular cylinder.

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